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PARETO CATASTROPHE THEORY

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1. INTRODUCTION

In mathematical economics, we often encounter the problem to optimize several functions simultaneously under restriction conditions. Pareto [1] proposed a notion of optimality for several functions, which is now called Pareto optimality. Smale [2] investigated the criteria for Pareto optimality. Hettich and Twente [3] reformulated Pareto optimality and proved that Pareto optimality is equivalent to the optimality of a function under some restriction conditions (in the ordinary sense).

In this paper, we interpret this situation as giving functions defined on manifolds with corners and optimality problem for such functions. We construct a field of functions, which will serve as a model in the framework of catastrophe theory [4] [5] [6].

2. PARETO OPTIMALITY AND FUNCTIONS ON MANIFOLDS WITH CORNERS

Let $p=(p_1,p_2,\ldots,p_n)$ and $q=(q_1,q_2,\ldots,q_n)$ be points in R^n. We denote $p \leqq q$ (resp. $p < q$) if $p_i \leqq q_i$ (resp. $p_i < q_i$) for $i=1,2,\ldots,n$. A function f defined on a subset X in R^n is said to be smooth if f can be extended to a smooth function defined on some neighbourhood of X. A closed subset X in R^n, with induced topology, is called a manifold with corners of dimension m if any point $x \in X$ has a neighbourhood of X which is diffeomorphic to some open set of the first quadrant $Q=\{y \in R^m \mid 0 \leqq y\}$ of R^m. Let $f : R^n \to R^1$ be smooth mapping. Let Z be the subset of R^n defined by $f(x) \leqq 0$ and $g(x)=0$.

Assume that Z is a manifold with corners of dimension $n-1$. Let $u : Z \to R^m$ be a smooth mapping defined on Z and that $u(z) > 0$ for every $z \in Z$.

<u>DEFINITION</u> A point $z_0 \in Z$ is said to be Pareto optimal on Z if there is no $z \in Z$ such that $u(z) \geqslant u(z_0)$ and $u(z) \neq u(z_0)$.

<u>DEFINITION</u> A point $z_0 \in Z$ is said to be Strong Pareto optimal on Z if there is no $z \in Z$ such that $u(z) \geqslant u(z_0)$ and $z \neq z_0$.

<u>DEFINITION</u> A point $z_0 \in Z$ is said to be local Pareto optimal (resp. local strong Preto optimal) if there is a neighbourhood U_0 of z_0 in R^n such that z_0 is Pareto optimal (resp. strong Pareto optimal) on $U_0 \cap Z$.

For a point $z_0 \in Z$, define a function v_{z_0} on a closed set V_{z_0} as follows. Let $w_i = (u^i(z_0))^{-1}$. Numbers w_i are positive. Define the closed set $V_{z_0} \subset Z \times R$ by

$$V_{z_0} = \left\{ (z,v) \in Z \times R \;\middle|\; v \leqslant w_i u^i(z),\ i=1,2,\ldots,m \right\}.$$

The function $v_{z_0} : V_{z_0} \to R$ is defined by $v_{z_0}(z,v) = v$.

For each z in Z, we can define a function $v_z : V_z \to R$ by taking z as z_0. We have a family of functions v_z with parameter space Z. Generically (i.e. for generic u and z), V_z is a manifold with corners. Hettich and Twente [3] gave the following reformulation of Pareto optimality.

<u>PROPOSITION</u>

If $z_0 \in Z$ is Pareto optimal (resp. local Pareto optimal, strong Pareto optimal, local strong Pareto optimal), then the point $(z_0, 1)$ is a maximal point (resp. local maximal point, strong maximal point, local strong maximal point) of the function $v_{z_0} : V_{z_0} \to R$, and vice versa.

They gave also the first and second order characterisations for them.

3. THE FUNCTION FIELD OVER Z

In the preceeding section we constructed a field of functions. For each point $z \in Z$, we have a function v_z defined on a closed set V_z. If $(z,1) \in V_z$ maximizes the function v_z, then the point $z \in Z$ is Pareto optimal. Define a set $V \subset Z \times R \times Z$ by

$$V = \left\{ (z_1,v,z_2) \in Z \times R \times Z \;\middle|\; (z_1,v) \in V_{z_2} \right\}.$$

We denote by pr_2 the projection of V onto the last factor Z, i.e. $pr_2(z_1,v,z_2) = z_2$. Let S denote the set of points $(z_1,v,z_2) \in V$ such that (z_1,v) maximizes the function $v_{z_2} : V_{z_2} \to R$. Let D denote the diagonal set defined by $D = \left\{ (z_1,v,z_2) \in V \;\middle|\; z_1=z_2,\ v=1 \right\}$. Let $P' = S \cap D$ and $P = pr_2(P')$. By definition the set P is the set of Pareto optimal points.

Let us examine some examples. Consider the economy where there are two consumers p_0, p_1 and two commodities A and B. Let $I_1 = [0,a]$ and $I_2 = [0,b]$. Let Z be the rectangular domain $I_1 \times I_2$ in R^2. The point $(x_1,x_2) \in Z$ represents the situation where consumer p_0 posesses the quantity x_1 of commodity A and x_2 of B and p_1 posesses $(a-x_1)$ of A and $(b-x_2)$ of B, so that the sum of commodities are a and b respectively. Let u_1 be the utility function of the first consumer which we assume to be positive on Z and differentiable and monotone increasing in x_1 and x_2 and to have level curves as depicted in fig.1. Let u_2 be the utility function of the second consumer similar to u_1 (see fig.2).

If the level curves are either strictly concave or strictly convex at least in the interior of Z, the Pareto set consists of a curve of points where level curves have common tangent line (see fig.3).

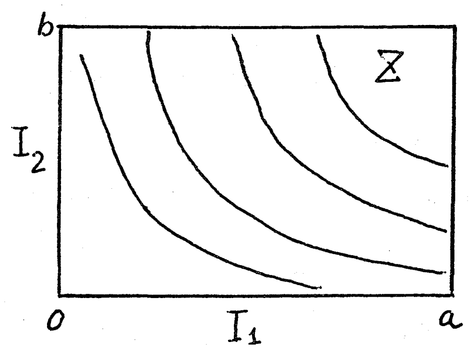


fig.1.
level curves of u_1

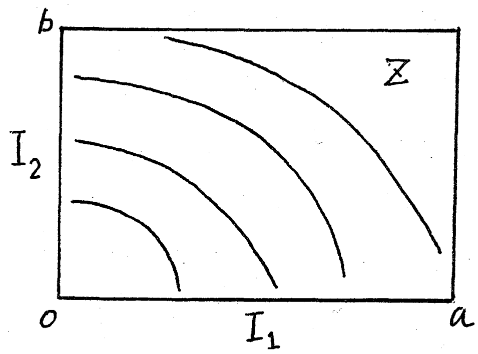


fig.2.
level curves of u_2

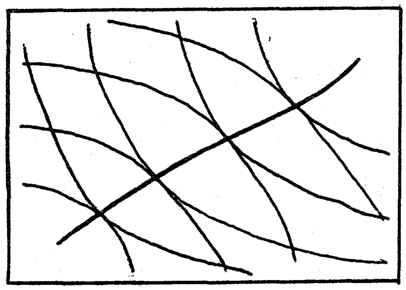


fig.3.
The Pareto set

For a point $z_0 = (x_1, x_2)$ in Z, the set V_z is defined by

$$V_{z_0} = \left\{ (z,v) \in Z \times R \;\middle|\; v \leqslant \frac{u_1(z)}{u_1(z_0)} \quad \text{and} \quad v \leqslant \frac{u_2(z)}{u_2(z_0)} \right\}$$

(see fig.4).

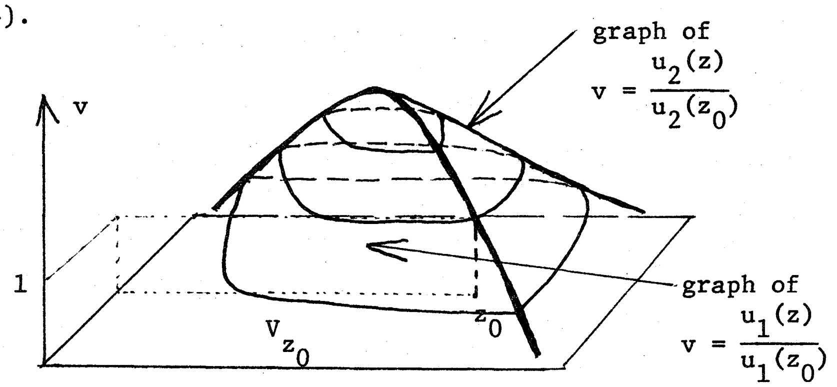


fig.4. the set V_{z_0}

The set V_{z_0} has an edge over the curve $u_2(z)u_1(z_0) = u_1(z)u_2(z_0)$. If another point z_1 is on this curve, the set V_{z_1} will have the same form as V_{z_0} multiplied by the constant $\frac{u_2(z_1)}{u_2(z_0)}$ in the v direction. The points which maximize v_{z_0} and v_{z_1} respectively have same coordinate (x_1,x_2). If we assume that the trade between two consumers is done only if the ratio of the values of two utility functions does not change and the values $u_1(z)$ and $u_2(z)$ increase, then the representative point z moves along this curve. Take this curve as an inner space and take another curve transversal to this curve as a control parameter. We obtain (at least locally) a model for catastrophe theory with potential function v.

In a rather sophisticated construction, we obtain a model for elementary catastrophe theory (in rather generalized sense) as follows.

We can define a smooth map U of Z into the real projective space P^{m-1} by composing the mapping u and the canonical projection of $R^m - \{0\}$ onto P^{m-1}. Foliate the set Z by taking inverse image by U of a point in P^{m-1} as a leaf. The obtained foliation may have singularities. The model is constructed by taking each leaf as the inner space and the point in P^{m-1} representing control parameters.

In this formulation, the Pareto set corresponds to a portion of the slow manifold of the model where potential function is maximized (see fig.5).

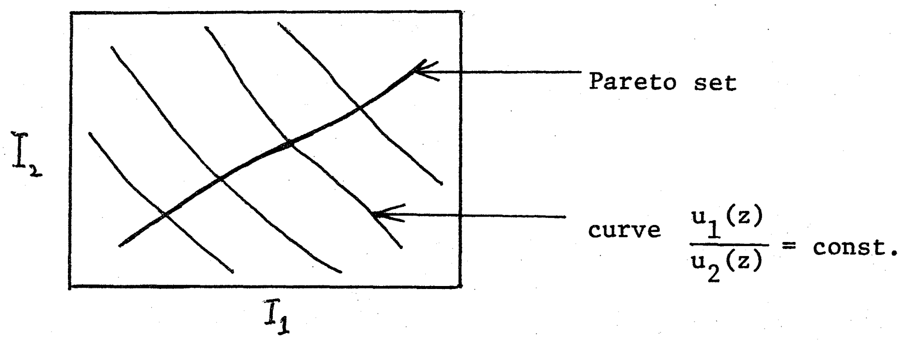


fig.5.

4. CUSP CATASTROPHE AND PARETO CATASTROPHE

Let us consider the situation where utility functions u_1 and u_2 changes their form smoothly as some exterior factor, say t , for example time, varies. Suppose that at $t = t_0$ we have utility functions as in the preceeding section and that at $t = t_1$ the level curves of u_1 and u_2 takes the contour as depicted in fig.6.

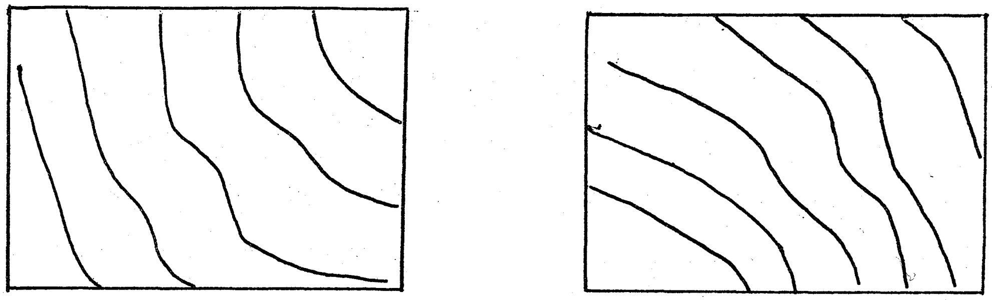


fig.6.
level curves of u_1 and u_2

At $t = t_1$, the convexity of level curves are lost. The foliated model at $t = t_1$ is illustrated in fig.7.

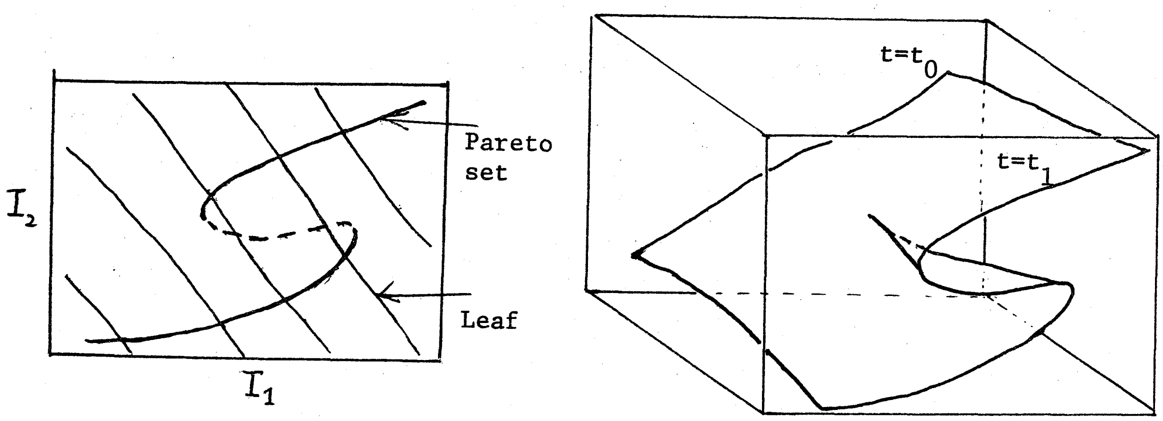
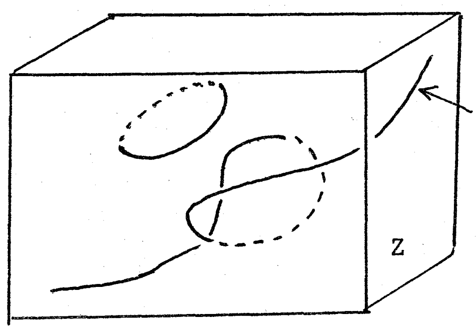


fig.7.
the space Z foliated at $t=t_1$ with Pareto set

Putting together the model for t $[t_0,t_1]$, we obtain a cusp-like Pareto catastrophe model.

5. OTHER CASES

If the dimension of the commodity space is three or more, and the number of consumers is still two, then the codimension of the inner space is one and the Pareto set is one dimensional(in the generic sense).



If the number of utility functions is three or more, say N, then the codimension of the inner space is N-1 and the dimension of Pareto set is N-1 generically.

It will be interesting to investigate the model where the effect of the corners of V_z has something to do with the Pareto set. For example, consider the following situation.

We have two commodities and three consumers. At some point z, we suppose that the configuration of V_z is as depicted in fig.9.

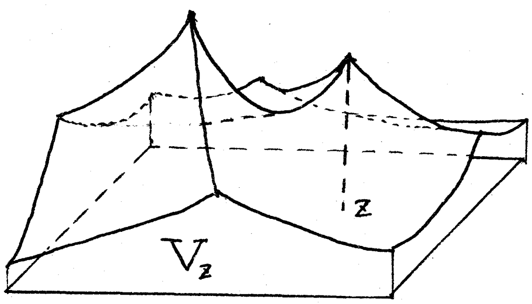


fig.9.

V_z has two peaks. If these two peaks are sharp as the Everest, the points near the points at the foot of these peaks are local Pareto optimal.

Global Pareto optimality is determined by the height of the peak. If, by the change of exterior parameter t, the highest peak becomes lower than the other, a Pareto catastrophe occurs.

Even if we impose the 'delay rule' instead of 'Maxwell's convention', we will observe the disappearance of Pareto optimal points in the situation illustrated in fig.10.

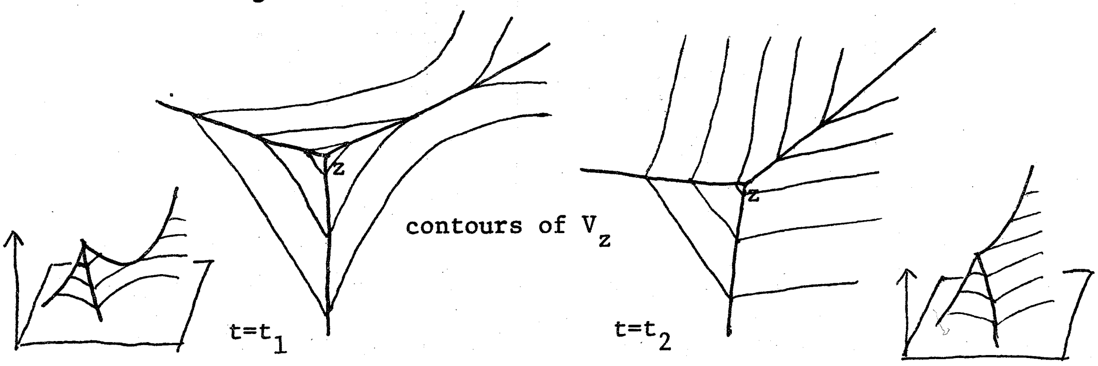


fig.10.
disappearance of local Pareto optimal point

Such a situation is seen even without the intervention of exterior control parameter. In this case, the Pareto set has a boundary and the points situated one of the two sides of the boundary are local Pareto optimal and the points situated in the other side are not local Pareto optimal.

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